H(infini)-feedback design for linear systems subject to input saturation

Yacine Chitour, Sami Tliba

To cite this version:

Yacine Chitour, Sami Tliba. H(infini)-feedback design for linear systems subject to input saturation. 2011. hal-00578908

HAL Id: hal-00578908

https://hal.science/hal-00578908

Preprint submitted on 23 Mar 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

$\mathcal{H}_\infty-$feedback design for linear systems subject to input saturation

Yacine Chitour* Sami Tliba*

February 27, 2011

1 Introduction

In this paper, we study linear systems subject to input saturation, i.e. systems of the type

$$(\Sigma)_{sat} \quad \dot{x} = Ax + b\sigma(u), \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ and $\sigma : \mathbf{R} \to \mathbf{R}$ is a saturation function whose prototype is the standard saturation function $\sigma_0(s) = \frac{s}{\max(1,|s|)}$. In this paper, we deliberately focus on single input systems, eventhough several results quoted below hold true for muti-inputs as well. The saturation-free linear system associated to $(\Sigma)_{sat}$ is naturally defined as

$$(\Sigma)_L \quad \dot{x} = Ax + bu, \tag{2}$$

Our goal regards robustness issues associated to the (global asymptotic) stabilization to the origin of systems (1). More precisely, assume that there exists an non empty set $\mathcal{K}$ of static feedback laws (i.e. $u = k(x)$) so that

(a) for each feedback law $k \in \mathcal{K}$, the closed loop system is GAS with respect to the origin;

(b) the L_2-gain associated to each $k \in \mathcal{K}$ is finite, i.e.

$$\gamma_k^{sat} := \sup_{d \in L_2(\mathbf{R}_+,\mathbf{R})} \frac{\|x_d\|_2}{\|d\|_2} < \infty,$$

where x_d is the trajectory of $\dot{x} = Ax + b\sigma(k(x) + d)$ with initial condition $x(0) = 0$.

Note immediately that if k is a linear feedback, then $\gamma_k^{sat} \geq S\gamma_k^L$, where γ_k^L is the L_2-gain associated to k for the linear system $(\Sigma)_L$ and S is a positive constant only depending on σ. Then, the issue we address is the following

$$\text{Estimate } \inf_{k \in \mathcal{K}} \gamma_k^{sat}, \tag{3}$$

* Laboratoire des Signaux et Systèmes, Supélec, 3, Rue Joliot Curie, 91192 Gif s/Yvette, France and Université Paris Sud, Orsay, yacine.chitour,sami.tliba@lss.supelec.fr

eventually in terms of L_2-gains of $(\Sigma)_L$. In particular, if (A,b) is controllable, recall that $\inf_{k \text{ linear}} \gamma_k^L = 0$. Our ultimate concern will be to achieve the same performance in the presence of saturation.

At the starting point of our study, (A,b) must be at least stabilizable, and by restricting the state space to the controllable space, we can (and will) assume, with no loss of generality, that the pair (A,b) is controllable. Moreover, it is well-known that (1) is stabilizable if and only if the eigenvalues of A must have non positive real part. Most delicate issues arise when the spectrum of A lies on the imaginary axis. The stabilization issue is already non trivial except for two dimensional systems which can be stabilized by linear feedbacks $u = k^T x$. However, it was proved by Sussman and Yang ([11] and also Fuller []) that the nth-integrator, $n \geq 3$ cannot be stabilized by linear feedbacks $u = k^T x$. Thanks to Teel [12] and Sussmann, Yang and Sontag [10], general and explicit static feedbacks were first constructed using nested saturations. However, as soon as the feedback requires at least two nested saturations, the corresponding L_2-gain is infinite, eventhough one can obtain $\mathcal{L}_2$ robustness results for these feedbacks at the price of a non linear concept of gain (cf. [13]). One should also mention the construction of another type of feedbacks based on "minimal" ellipsoids and due to Megretsky (cf. [9]). The evaluation of the L_2-performances of these feedbacks is still an open problem. Finally, let us mention that, in the context of semi-global stabilization, the L_2-performances of linear systems with saturations were resolved by Z. Lin and his coworkers, using a low-and-high gain design technique (cf. [5] and references therein).

At the light of the above discussion, a reasonable chance to address the issue of estimating $\inf_{k \in \mathcal{K}} \gamma_k^{sat}$ only occurs when the family of stabilizing keedbacks $\mathcal{K}$ is large enough. To the best of the authors knowledge, this is the case for particular cases only, besides the use of nested saturations. For instance, when A is skew-symmetric (and more generally marginally stable), it was proved that the feedbacks $u = -rb^T x$, $r > 0$ verify both Items (a) and (b) (cf. [7]) but $\inf_{r>0} \gamma_r^{sat} > 0$, where γ_r^{sat} is the associated L_2-gain. This is obtained by simply comparing γ_r^{sat} with the corresponding γ_r^L. However, for 2D and 3D systems and increasing saturation functions, in [1], the family of stabilizing feedbacks $\mathcal{K}$ was enlarged to at least all the linear feedbacks for $(\Sigma)_L$ and it was also proved that the L_∞-gains for these feedbacks were finite. It is not difficult to see that the same holds true for the L_2-gains but no gain attenuation result seems straitgthforward from the results of [1].

In the present paper, we return to the case of the 2D oscillator subject to input saturation and we determine an appropriate set $\mathcal{K}$ of linear feedbacks so that $\inf_{k \in \mathcal{K}} \gamma_k^{sat} = 0$.

We next deal with the case of the double integrator subject to input saturation i.e. $\ddot{x} = -\sigma(u)$, where $x \in \mathbf{R}$. Eventhough the latter system is stabilizable by means of linear feedbacks, it was shown in [7] that the corresponding L_2-gains are infinite for the main output map $y = x$. The same conclusion was reached for the output map $y = \dot{x}$ in [6] but it was proved in [3] that the L_2-gain for the output map $y = \ddot{x}$ was finite, partially solving a Problem 36 as posted in [2]. Still in [3], was provided a class of nonlinear feedbacks having a finite L_2-gain for the output map $y = x$. In this paper, we show how to select a subset $\mathcal{K}$ of this class of feedbacks so that to get that $\inf_{k \in \mathcal{K}} \gamma_k^{sat} = 0$.

As future works, several directions seem interesting. The first one consists in extending first the present results to the general case of A skew-symmetric. We conjecture that linear feedbacks should be enough to get $\inf_{k \in \mathcal{K}} \gamma_k^{sat} = 0$. It is definitely more difficult to reach the same results for marginally instable A and we suspect that new (non linear) feedbacks must be deviced for that purpose. Another line of research would be to investigate attenuation of incremental gains since partial results on that issue are available for A skew-symmetric ([8]).

2 Notations

We use L_2^+ to denote the Hilbert space $L_2(\mathbf{R}_+, \mathbf{R})$ and, for $d \in L_2^+$, we use $\|d\|_2$ to denote the corresponding L_2-norm of d.

Definition 1 *We call $\sigma : \mathbf{R} \to \mathbf{R}$ a (normalized) saturation function (or an S-function) if for all $t, t' \in \mathbf{R}$*

(i) $|\sigma(t) - \sigma(t')| \leq \inf(1, |t - t'|)$;

(ii) $|\sigma(t) - t| \leq t\sigma(t)$.

Note that (i) is equivalent to the fact that σ is bounded and globally Lipschitz. On the other hand, (ii) is equivalent to

$$\sigma'(0) = 1,\ t\sigma(t) > 0 \text{ for } t \neq 0,\ \liminf_{|t| \to \infty} |\sigma(t)| > 0,\ \limsup_{t \to 0} \frac{\sigma(t) - t}{t^2} < \infty.$$

For instance, arctan, tanh and σ_0 are examples of saturation functions.

We use $\Sigma(s)$ to denote, for $s \in \mathbf{R}$ the function defined by $\int_0^s \sigma(t)dt$. Note that $\Sigma(s) > 0$ for $s \neq 0$ and if σ is odd then Σ is an even function.

We also recall that for a linear system $\dot{x} = Ax + bu$, with (A, b) controllable, one has $\inf_{k \text{ linear}} \gamma_k^L = 0$ where γ_k^L is the L_2-gain associated to the output map $u \mapsto x$ and a stabilizing linear feedback k. Indeed, to see that, it is enough to prove it for the Brunovsky form associated to $\dot{x} = J_n x + be_n$ where J_n is the n-th Jordan block and $e_n = (0 \cdots 0\ \ 1)^T$. For the latter system, a direct computation using the family of dilations $D_\mu := \mathrm{diag}(\mu^{n-i})_{1 \leq i \leq n}$, $\mu > 0$ yields the conclusion (see also Remark 4.3).

3 Oscillator in dimension two

We consider the linear control system subject to input saturation given by

$$(\Sigma)_{sat} \quad \dot{x} = A_0 x - e_2 \sigma(u), \tag{4}$$

where $A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and σ is a real valued-function of "saturation" type.

The corresponding saturation free system is given by

$$(\Sigma)_L \quad \dot{x} = A_0 x - e_2 u. \tag{5}$$

According to the Routh-Hurwitz criterion, the linear stabilizing feedbacks for $(\Sigma)_L$ are row vectors $(k_1\ k_2)$ with $k_1 < 1$ and $k_2 > 0$. It was proved in [1] that these feedbacks also render $(\Sigma)_{sat}$ globally asymptotically stable with respect to the origin.

Proposition 3.1 (cf. [1]) *Let $\mathcal{S}$ be the subset of $\mathbf{R}^2$ whose elements are the row vectors $(k_1\ k_2)$ with $k_1 < 1$ and $k_2 > 0$. Then, for every $K \in \mathcal{S}$, the saturated system*

$$\dot{x} = A_0 x - e_2 \sigma(k_1 x_1 + k_2 x_2),$$

is globally asymptotically stable with respect to the origin.

Proof. Consider the Lyapunov function given by

$$V(x_1, x_2) = \|x\|^2 + \frac{2k_1}{k_1^2 + k_2^2} \Sigma(k_1 x_1 + k_2 x_2).$$

Then, the derivative of V along trajectories of $\dot{x} = A_0 x - e_2 \sigma(k_1 x_1 + k_2 x_2)$ is equal to

$$\dot{V} = -\frac{2k_2}{k_1^2 + k_2^2} \xi \sigma(\xi) \big(1 - k_1 \frac{\sigma(\xi)}{\xi}\big),$$

where $\xi := k_1 x_1 + k_2 x_2$. Since $k_1 < 1$ and $0 < \frac{\sigma(\xi)}{\xi} \leq 1$ for $\xi \in \mathbf{R}$, one gets that $\dot{V} \leq -\frac{2k_2(1-k_1)}{k_1^2+k_2^2} \xi \sigma(\xi)$ and then concludes by applying Lasalle's principle.

■

For every linear stabilizing feedback $K \in \mathcal{S}$ $(\Sigma)_L$, the linear gain γ_K^L associated to the output map $y = x$ is defined as

$$\gamma_K^L := \sup_{d \neq 0} \frac{\|x_d\|_2}{\|d\|_2}, \tag{6}$$

where $d \in L_2(\mathbf{R}_+, \mathbf{R})$ and x_d is the unique solution of $\dot{x} = (A_0 - e_2 K^T)x - e_2 d$ with $x(0) = 0$. It is well-known that γ_K^L is finite and verifies the classical Riccati criterium given next (cf. [4]): for every $\gamma > \gamma_K^L$, there exists a unique symmetric definite positive matrix P solution of the following Ricatti equation,

$$P(A_0 - e_2 K^T) + (A_0 - e_2 K^T)^T P + \frac{1}{\gamma^2} P e_2 e_2^T P + I_2 = 0, \tag{7}$$

and such that $A + \frac{1}{\gamma^2} e_2 e_2^T P$ is Hurwitz. Moreover, the infimum of γ_K^L over all the linear stabilizing feedback K is equal to zero.

The goal of this note is to first determine a large set $\mathcal{S}' \subset \mathcal{S}$ of linear stabilizing feedbacks K for $(\Sigma)_{sat}$ having a finite (saturated) gain γ_K^{sat} (associated to the output map $y = x$) i.e.,

$$\gamma_K^{sat} := \sup_{d \neq 0} \frac{\|x_d\|_2}{\|d\|_2} < \infty, \tag{8}$$

where $d \in L_2(\mathbf{R}_+, \mathbf{R})$ and x_d is the unique solution of $\dot{x} = A_0 - e_2 \sigma(K^T x + d)$ with $x(0) = 0$. Then, we will prove that the infimum of γ_K^{sat} over all the linear stabilizing feedback K of $\mathcal{S}'$ is equal to zero.

To proceed, we operate a linear change of variable for each feedback $K \in \mathcal{S}$. We consider now the following saturated system

$$(\Sigma_{sat})_K \quad \dot{x} = A_0 - b\sigma(k e_2^T x + d), \tag{9}$$

where $b := \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix}$ and $k > 0$. Here we use c_θ and s_θ to denote respectively $\cos\ \theta$ and $\sin\ \theta$. The corresponding linear system (i.e., saturation free) is equal to

$$(\Sigma_L)_K \quad \dot{x} = (A_0 - kbe_2^T)x - bd, \tag{10}$$

and the stability condition reads

$$s_\theta > 0 \text{ and } 1 + kc_\theta > 0.$$

Moreover, the Ricatti equation (7) becomes now

$$P(A_0 - kbe_2^T) + (A_0 - kbe_2^T)^T P + \frac{1}{\gamma^2} Pbb^T P + I_2 = 0, \tag{11}$$

In the sequel, we will use repeatedly the notation R_α, $\alpha \in \mathbf{R}$, to denote the rotation of angle α, i.e.,

$$R_\alpha := \begin{pmatrix} c_\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}.$$

In particular, $b = R_\alpha e_1$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We next sketch the rest of the argument. We will first compute the linear gain γ_L defined in (6) for $\theta < \pi/2$ and close to $\pi/2$ and k large enough. We then compute the symmetric positive definite matrix P defined in (7) as well as the vector Pb corresponding to the above mentionned of k and θ. In a last step, we provide an upper bound of γ_{sat} in terms of γ_L and conclude.

Lemma 3.2 *Assume that $k > 0$, $\theta \in (0, \pi/2)$ and $4 + 4kc_\theta + k^2(c_\theta^2 - s_\theta^2) < 0$. Then the linear gain γ_K^L of $\dot{x} = (A_0 - kbe_2^T)x - bd$, associated to the output map $y = x$, is equal to*

$$\gamma_K^L = \frac{1}{1 + kc_\theta}. \tag{12}$$

In particular, the conditions on k and θ in the above statement imply that $\theta \in (\pi/4, \pi/2)$. Moreover, the subscript K stands for a choice of (k, θ) in the range defined previously.
Proof. We proceed by a standard computation of the H_∞-gain of the linear system $\dot{x} = (A_0 - kbe_2^T)x - bd$ associated to the output map $y = x$.

The closed-loop transfer mapping $G(s)$ is given by

$$G(s) = \frac{1}{s^2 + ks_\theta s + 1 + kc_\theta} \begin{pmatrix} -c_\theta s + s_\theta \\ -s_\theta s - c_\theta \end{pmatrix}.$$

Then one has the standard formula (cf. for instance [4]) asserts that

$$\gamma_K^L = \|G(s)\|_\infty = \sup_{\omega \in \mathbf{R}} \bar{\sigma}(G(j\omega)),$$

where $\bar{\sigma}(G(j\omega))$ is used to denote the maximum singular value of $G(j\omega)$ for $\omega \in \mathbf{R}$. One easily gets that

$$\bar{\sigma}(G(j\omega)) = \sqrt{\frac{1 + \omega^2}{(\omega^2 - 1 - kc_\theta)^2 + (ks_\theta \omega)^2}}.$$

One has therefore to determine the maximum of the function $f\mathbf{R}_+ \to \mathbf{R}_+$ defined by

$$f(X) = \frac{1+X}{(X-1-kc_\theta)^2 + (ks_\theta X)^2}.$$

A straitghforward computation shows that, if $4 + 4kc_\theta + k^2(c_\theta^2 - s_\theta^2) < 0$, then the maximum is reached at $X = 0$ and is equal to $\frac{1}{(1+kc_\theta)^2}$.

■

Lemma 3.3 *Assume that $\theta \in (0, \pi/2)$ and $4 + 4kc_\theta + k^2(c_\theta^2 - s_\theta^2) < 0$. Then, the unique symmetric positive definite matrix P, solution of the Riccati equation, defined in (11) and such that $A + \frac{1}{\gamma^2} e_2 e_2^T P$ is Hurwitz, is equal to*

$$P = \frac{1}{\Delta} R_{\theta+\varphi} \Big[\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + O(\frac{1}{k^2}) \Big] R_{-(\theta+\varphi)}, \tag{13}$$

where

$$\tan\ \varphi = -\frac{1}{ks_\theta}, \quad \Delta = 2(1+kc_\theta) - \frac{c_\theta}{s_\theta}(1+\frac{c_\theta}{s_\theta}) + O(\frac{1}{k}), \tag{14}$$

and

$$p_1 = \frac{c_\theta}{s_\theta} + \frac{1}{ks_\theta}, \ p_2 = \frac{2c_\theta}{ks_\theta^2}, \ p_3 = 2kc_\theta - (1+\frac{c_\theta}{s_\theta}). \tag{15}$$

Finally, one has

$$Pb = \frac{c_\theta}{2s_\theta(1+kc_\theta)} R_\theta \begin{pmatrix} 1 \\ 2 \end{pmatrix} + O(\frac{1}{k^2}). \tag{16}$$

In the above equation, $O(\cdot)$ stands the standard "Big O" notation as k or kc_θ tends to infinity. All constants involved in this notation do not depend on k or $\theta \in (\pi/4, \pi/2)$.

Proof. . For $\gamma > \gamma_K^L$, the symmetric solution of the Ricatti equation $P(\gamma)$ is positive definite. Define $Q(\gamma) := P^{-1}(\gamma)$. Then $Q(\gamma)$ is the symmetric positive definite solution of the equation

$$(Q+A)(Q+A)^T = AA^T - \frac{1}{\gamma^2} bb^T. \tag{17}$$

Assume that the above equation admits a symmetric positive definite solution $\overline{Q}$ for $\gamma = \gamma_K^L$. By a simple continuity argument, one gets that $P(\gamma)$ tends to $\overline{Q}^{-1}$ as γ tends to γ_K^L, for fixed k and θ in the admissible range. Thus $P(\gamma_K^L)$ is well defined (as symmetric positive definite solution of (11) and equal to $\overline{Q}^{-1}$).

We can therefore solve (17) for $\gamma = \gamma_K^L$ and, as we will notice ultimately, the symmetric solution to be found will be positive definite.

We first find that

$$AA^T - \frac{1}{(\gamma_K^L)^2} bb^T = X^2 R_{\theta+\varphi} e_1 e_1^T R_{-(\theta+\varphi)},$$

where $X = \sqrt{1 + ks_\theta}$ and φ is the angle defined by $\tan\ \varphi = -\frac{1}{ks_\theta}$ (here and after k is supposed to be large with respect to 1). Set $S := R_{\theta+\varphi} e_1 e_1^T R_{-(\theta+\varphi)}$.

Then, there exists a rotation R_η such that

$$\overline{Q} + A = XSR_\eta = XR_{\theta+\varphi} e_1 e_1^T R_{-(\theta+\varphi)+\eta}.$$

Since $\overline{Q}$ is symmetric, one has

$$A - A^T = X(SR_\eta - R_{-\eta}S) = XR_{\theta+\varphi}(e_1e_1^TR_\eta - R_{-\eta}e_1e_1^T)R_{-(\theta+\varphi)}.$$

The matrix in parentheses in the last expression is skew-symmetric and therefore commutes with the rotation $R_{\theta+\varphi}$. That implies that the last equation reduces to

$$A - A^T = X(e_1e_1^TR_\eta - R_{-\eta}e_1e_1^T).$$

A simple computation then yields

$$s_\eta = \frac{2+kc_\theta}{X}. \tag{18}$$

On the other hand, one has

$$\overline{Q} = -\frac{A+A^T}{2} + X\frac{SR_\eta + R_{-\eta}S}{2}.$$

After some computations, one gets

$$\overline{Q} = R_{\theta+\varphi}\left[k(s_\theta R_{-\eta}e_1e_1^TR_\eta + \frac{c_\theta}{2}R_{-\eta}(e_1e_2^T + e_2e_1^T)R_\eta + \frac{X}{2}(e_1e_1^TR_\eta + R_{-\eta}e_1e_1^T)\right]R_{-(\theta+\varphi)},$$

which in turn yieds that

$$\overline{Q} = R_{\theta+\varphi}\begin{pmatrix} q_1 & q_2 \\ q_2 & q_3 \end{pmatrix}R_{-(\theta+\varphi)},$$

where

$$\begin{aligned} q_1 &= ks_\theta c_\varphi^2 + kc_\theta c_\varphi s_\varphi + Xc_\eta, && (19)\\ q_2 &= -ks_\theta c_\varphi s_\varphi + \frac{kc_\theta}{2}(c_\varphi^2 - s_\varphi^2) - \frac{Xs_\eta}{2}, && (20)\\ q_3 &= -kc_\theta c_\varphi s_\varphi - ks_\theta s_\varphi^2. && (21) \end{aligned}$$

By taking into account the definitions of X and the angles φ and η, one gets

$$\begin{aligned} q_1 &= k(s_\theta - \frac{c_\theta}{s_\theta})\frac{k^2s_\theta^2}{1+k^2s_\theta^2} + ks_\theta\sqrt{(}1 - \frac{1+2kc_\theta}{k^2s_\theta^2}) = 2ks_\theta - (1+\frac{c_\theta}{s_\theta}) + O(\frac{1}{k^2}), && (22)\\ q_2 &= -\frac{c_\theta}{ks_\theta^2} - \frac{1+kc_\theta}{1+k^2s_\theta^2} = -\frac{2c_\theta}{ks_\theta^2} + O(\frac{1}{k^2}), && (23)\\ q_3 &= (\frac{c_\theta}{s_\theta} + \frac{1}{ks_\theta})\frac{k^2s_\theta^2}{1+k^2s_\theta^2} = \frac{c_\theta}{s_\theta} + \frac{1}{ks_\theta} + O(\frac{1}{k^2}). && (24) \end{aligned}$$

Setting $\Delta := \det Q$, one gets at once (14) and then (13) and (15) for an analytical expression of P.

Recalling that $b = R_\theta e_1$, one gets that

$$Pb = \frac{c_\varphi}{\Delta}R_{\theta+\varphi}\begin{pmatrix} q_3 & -q_2 \\ -q_2 & q_1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{ks_\theta} \end{pmatrix}.$$

Using further that $c_\varphi R_\varphi = \frac{k^2s_\theta^2}{1+k^2s_\theta^2}(I_2 - \frac{1}{ks_\theta}A_0)$, one derives (16).

$\blacksquare$

We can now address the question of $\mathcal{H}_\infty-$feedback design for $(\Sigma)_{sat}$. In order to access to upper bounds of γ_K^{sat}, we resort to a dissipative inequality involving an appropriate storage function. More precisely, assume that $\theta \in (0, \pi/2)$, $4 + 4kc_\theta + k^2(c_\theta^2 - s_\theta^2) < 0$ and k is large. We prove the following theorem.

Theorem 3.4 *There exists $\theta_0 \in (\pi/4, \pi/2)$ and $k_0 > 0$ such that, for every $k \geq k_0$, one has*

$$\gamma_K^{sat} \leq 30\gamma_K^L. \tag{25}$$

In particular, for that choice of K (i.e., choice of (k, θ_0) with $k \geq k_0$), $(\Sigma)_{sat}$ has a finite $\mathcal{L}_2$-induced gain for the output $y = x$ and the infimum of these gains over all the stabilizing feedbacks is equal to zero.

Proof. . Along trajectories of $((\Sigma_{sat})_K)$ given in (9), consider the Lyapunov function W defined by

$$W(x) = x^T P x + \frac{C_0}{3}\|x\|^3, \tag{26}$$

where P is the symmetric positive definite matrix provided by Lemma 3.3 and $C_0 > 0$ is a constant to be determined. Set $\xi := ke_2^T x + d$. Then one has

$$(x^T P x)^{\cdot} = -\|x\|^2 - \frac{1}{\gamma_L^2}(x^T P b)^2 - 2d(x^T P b) + 2(x^T P b)(\xi - \sigma(\xi)). \tag{27}$$

It is the last term of the above equality which cannot be controlled, neither by the term "$-\|x\|^2$" nor by a completion of squares involving $-\frac{1}{\gamma_L^2}(x^T P b)^2$. This is why one must add another term to $x^T P x$ in order to achieve to a dissipation inequality.

Consider the following

$$(\|x\|^3/3)^{\cdot} = \|x\| x^T \dot{x} = -\|x\|(c_\theta e_1^T x \sigma(\xi) + \frac{\xi - d}{k} s_\theta \sigma(\xi)). \tag{28}$$

We next take $C_0 = \frac{2k\|Pb\|}{s_\theta}$. The above equation yields

$$(\frac{C_0}{3}\|x\|^3)^{\cdot} \leq C_0 c_\theta \|x\|^2 + \frac{2\|Pb\|}{s_\theta}\|x\||d| - 2\|x\|\|Pb\|\xi\sigma(\xi). \tag{29}$$

Using Item (ii) in Definition 1, we can get rid of the last term of (27) with the last term of (29). One then gets

$$\dot{W} \leq -(1 - C_0 c_\theta)\|x\|^2 + \frac{2\|Pb\|}{s_\theta}\|x\||d| - \frac{1}{\gamma_L^2}(x^T P b)^2 + 2|d||x^T P b|. \tag{30}$$

By choosing k large enough, one has

$$C_0 c_\theta \leq \frac{3c_\theta}{s_\theta^2} \text{ and } \frac{2\|Pb\|}{s_\theta} \leq \frac{6}{ks_\theta^2}.$$

By choosing $\theta = \theta_0$ close enough to $\pi/2$ and independently of k, one also has

$$C_0 c_{\theta_0} \leq 1/2 \text{ and } \frac{2\|Pb\|}{s_{\theta_0}} \leq 7\gamma_K^L.$$

For the choice $\theta = \theta_0$ and for k large enough, (30) becomes

$$\dot{W} \leq -\frac{1}{2}\|x\|^2 + 7\gamma_K^L\|x\||d| - \frac{1}{(\gamma_K^L)^2}(x^T Pb)^2 + 2|d||x^T Pb|.$$

After completion of squares, the above inequality can be written as

$$\dot{W} \leq -\frac{1}{4}\|x\|^2 + (1 + 14^2)(\gamma_K^L)^2|d|^2. \tag{31}$$

It is then immediate to derive (25) from the above inequality.

■

Remark 3.5 The choice of the Lyapunov function W as defined in (26) was already performed in [7]. However, in the abovementionned reference, the symmetric positive definite matrix P was taken as solution of the Lyapunov equation $A^T P + PA = -I_2$. Moreover, the feedback was taken with the choice $k > 0$ and $\theta = 0$. The corresponding linear gain was always equal to one.

Remark 3.6 One of the critical inequalities obtained previously was $C_0 c_\theta := \frac{2kc_\theta\|Pb\|}{s_\theta} \leq \frac{3c_\theta}{s_\theta^2}$, which then allows to use the angle θ as an extra degree of freedom in order to get $C_0 c_\theta < 1$. It was therefore important to have a "good" upper bound of $\|Pb\|$. One can derive a first estimate of $\|Pb\|$ without an exact computation of P itself. Indeed, by taking the trace in (11), one deduces that

$$\|Pb\| \leq k(\gamma_K^L)^2 + \gamma_K^L\sqrt{(k\gamma_K^L)^2 - 2}.$$

Then the smallest value for $C_0 c_\theta$ is obtained as θ tends to $\pi/4$ and is equal to 4, which is not "good" enough for our purposes.

4 Double integrator

In this section, we consider the double integrator system subject to input satutation, i.e. the linear control system defined by

$$(DI)_{sat} \quad \dot{x} = J_2 x - e_2\sigma(u), \tag{32}$$

where $x = (x_1, x_2)^T$, J_2 is the 2D Jordan block, i.e. $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and σ is a real valued-function of "saturation" type.

In [3], non linear feedbacks were introduced to obtain finite L_2-gains. For that purpose, we define the class of $\mathcal{F}$-functions as follows:

Definition 2 *A function $F : \mathbf{R} \to \mathbf{R}$ is an $\mathcal{F}$-function if F is C^1, odd, $F'(0) = 0$ and there exist $r \geq 1$ such that, for $|x_2| \geq 1$, we have*

$$F'(x_2) \geq \sup\Big(3|x_2|, r\frac{F(x_2)}{x_2}\Big). \tag{33}$$

We now recall the result of [3] to be used later.

Proposition 4.1 *Let us consider the control system $(DI)_{sat}$ with σ an increasing saturation function, and the feedback $k_F(x)$ given by*

$$k_F(x) := x_1 + x_2 + F(x_2), \tag{34}$$

where F an $\mathcal{F}$-function. Then k_F is a feedback stabilizer for $(DI)_{sat}$ with finite L_2-gain for the output map $y = x$.

For instance $x_1 + x_2 + 3x_2|x_2|$ and $x_1 + x_2 + x_2^3$ are examples of k_F-feedbacks.

Then, the main result of this section is the following.

Theorem 4.2 *Consider the double integrator system subject to input satutation as defined by in (32) and a feedback k_F defined by some $\mathcal{F}$-function. For $\mu > 0$, consider the feedback*

$$k_F^\mu(x) := \mu^2 x_1 + \mu x_2 + F(\mu x_2). \tag{35}$$

If $\gamma(\mu)$ is used to denote the L_2-gain associated to k_F^μ for the output map $y = x$, then, for $\mu \geq 1$, one has

$$\gamma(\mu) \leq \frac{\gamma(1)}{\mu}, \tag{36}$$

where $\gamma(1)$ is L_2-gain associated to k_F for the output map $y = x$. As a consequence,

$$\lim_{\mu \to \infty} \gamma(\mu) = 0.$$

Proof. For $\mu > 0$, consider the feedback k_F^μ given in (35). Make the following change of variable and time

$$\begin{aligned} X_1(t) &= \mu^2 x_1(t/\mu), & (37)\\ X_2(t) &= \mu x_2(t/\mu), & (38)\\ V(t) &= u(t/\mu). & (39) \end{aligned}$$

The state $X := (X_1, X_2)^T$ verifies

$$\dot{X} = J_2 X - e_2 \sigma(k_F(X) + V).$$

According to Proposition 4.1, one has

$$\|X\|_2 \leq \gamma(1)\|V\|_2.$$

Since one has

$$\|X_1\|_2 = \mu^{5/2}\|x_1\|_2, \quad \|X_2\|_2 = \mu^{3/2}\|x_2\|_2, \quad \|V\|_2 = \mu^{1/2}\|u\|_2,$$

one concludes readily.

Remark 4.3 Actually, the above technique of dilations is standard and can be generalized as well for the n-th integrator subject to input saturation, in the case where there exists a stabilizing (static) feedback law with finite L_2-gain. It works as follows.

Assume that k is a stabilizing (static) feedback law with finite L_2-gain i.e.,

$$\gamma^{sat} := \sup_{d \in L_2(\mathbf{R}_+, \mathbf{R})} \frac{\|x_d\|_2}{\|d\|_2} < \infty,$$

where x_d is the solution of the perturbed system $\dot{x} = J_n x + e_n \sigma(k(x) + d)$ with $x(0) = 0$.

For $\mu > 0$, set $D_\mu = \operatorname{diag}(\mu^{n-i})_{1 \le i \le n}$. Then, one has

$$\mu D_\mu J_n D_\mu^{-1} = J_n, \quad D_\mu e_n = e_n.$$

Also set $k_\mu(\cdot) := k(\mu D_\mu \cdot)$ for $\mu > 0$. Then

$$\gamma_\mu^{sat} := \sup_{d \in L_2(\mathbf{R}_+, \mathbf{R})} \frac{\|z_d\|_2}{\|d\|_2} < \infty,$$

where z_d is the solution of the perturbed system $\dot{x} = J_n x + e_n \sigma(k_\mu(x) + d)$ with $x(0) = 0$. Indeed, $x_d^\mu(\cdot) := \mu D_\mu z_d(\cdot/\mu)$ is the solution of $\dot{x} = J_n x + e_n \sigma(k(x) + d(\cdot/\mu))$, with $x_\mu(0) = 0$. For $\mu \ge 1$, one immediately gets,

$$\mu^{3/2} \|z_d\|_2 \le \|x_d^\mu\|_2 \le \gamma^{sat} \|d(\cdot/\mu)\|_2 = \gamma^{sat} \mu^{1/2} \|d\|_2,$$

and then deduces that $\gamma_\mu^{sat} \le \gamma^{sat}/\mu$ and concludes readily.

∎

References

[1] Angeli D., Chitour Y. and Marconi L., "Robust Stabilization via Saturated Feedback," IEEE TAC, Vol 50, no 12, 2005, pp 1997-2014.

[2] Blondel V., Sontag E.D., Vidyasagar M. and Willems J., "Open problems in Mathematical Systems and Control Theory," Springer-Verlag, 1999.

[3] Chitour Y., esaim cocv

[4] Green M. and Limebeer D.J.N., "Linear Robust Control," Information and System Sciences Series, Prentice Hall, 1995.

[5] Hu T., and Lin Z., "Control Systems with Actuator Saturation: Analysis and Design," Birkhauser, Boston, 2001.

[6] Kao C.-Y., "On the $\mathcal{L}_2$-gain of a double integrator with feedback loop saturation nonlinearity," IEEE TAC, 46 (3), pp 501-504, 2001.

[7] Liu W., Chitour Y. and Sontag E. D., siam paper

[8] Liu W., Chitour Y. and Sontag E. D., IJRNLC paper

[9] Megretski, A. "$\mathcal{L}_2$ output feedback stabilization with saturated control," IFAC 96, pp 435-440, San Franciscco, California.

[10] Sussmann, H.J., Yang, Y. and Sontag E.D., "A general result on the stabilization of linear systems using bounded controls," IEEE TAC, 39 (12), pp 2411-2425, 1994.

[11] Sussmann H.J. and Yang Y.,

[12] Teel A.R., "Global stabilization and restricted tracking for multiple integrators with bounded controls," Systems and Control Letters, 18 (3), pp 165-171, 1992.

[13] Teel A.R., "On $\mathcal{L}_2$ performance induced by feedbacks with multiple saturations, " ESAIM COCV, 1, pp 225-240, 1996.